

Linear Coloring and Linear Graphs^{*}

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Abstract: Motivated by the definition of linear coloring on simplicial complexes, recently introduced in the context of algebraic topology [9], and the framework through which it was studied, we introduce the linear coloring on graphs. We provide an upper bound for the chromatic number $\chi(G)$, for any graph G , and show that G can be linearly colored in polynomial time by proposing a simple linear coloring algorithm. Based on these results, we define a new class of perfect graphs, which we call co-linear graphs, and study their complement graphs, namely linear graphs. The linear coloring of a graph G is a vertex coloring such that two vertices can be assigned the same color, if their corresponding clique sets are associated by the set inclusion relation (a clique set of a vertex u is the set of all maximal cliques containing u); the linear chromatic number $\lambda(G)$ of G is the least integer k for which G admits a linear coloring with k colors. We show that linear graphs are those graphs G for which the linear chromatic number achieves its theoretical lower bound in every induced subgraph of G . We prove inclusion relations between these two classes of graphs and other subclasses of chordal and co-chordal graphs, and also study the structure of the forbidden induced subgraphs of the class of linear graphs.

Keywords: Linear coloring, chromatic number, linear graphs, co-linear graphs, chordal graphs, co-chordal graphs, strongly chordal graphs, algorithms, complexity.

1 Introduction

Framework-Motivation. A *linear coloring* of a graph G is a coloring of its vertices such that if two vertices are assigned the same color, then their corresponding clique sets are associated by the set inclusion relation; a *clique set* of a vertex u is the set of all maximal cliques in G containing u . The linear chromatic number $\lambda(G)$ of G is the least integer k for which G admits a linear coloring with k colors.

Motivated by the definition of linear coloring on simplicial complexes associated to graphs, first introduced by Civan and Yalçın [9] in the context of algebraic topology, we define the linear coloring on graphs. The idea for translating their definition in graph theoretic terms came from studying linear colorings on simplicial complexes which can be represented by a graph. In particular, we studied the linear coloring on the independence complex $\mathcal{I}(G)$ of a graph G , which can always be represented by a graph and, more specifically, is identical to the complement graph \overline{G} of G in graph theoretic terms; indeed, the facets of $\mathcal{I}(G)$ are exactly the maximal cliques of \overline{G} . However, the two definitions cannot always be considered as identical since not in all cases a simplicial complex can be represented by a

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graph; such an example is the neighborhood complex $\mathcal{N}(G)$ of a graph G . Recently, Civan and Yalçın [9] studied the linear coloring of the neighborhood complex $\mathcal{N}(G)$ of a graph G and proved that, for any graph G , the linear chromatic number of $\mathcal{N}(G)$ gives an upper bound for the chromatic number of the graph G . This approach lies in a general framework met in algebraic topology.

In the context of algebraic topology, one can find much work done on providing boundaries for the chromatic number of an arbitrary graph G , by examining the topology of the graph through different simplicial complexes associated to the graph. This domain was motivated by Kneser's conjecture, which was posed in 1955, claiming that "if we split the n -subsets of a $(2n + k)$ -element set into $k + 1$ classes, one of the classes will contain two disjoint n -subsets" [16]. Kneser's conjecture was first proved by Lovász in 1978, with a proof based on graph theory, by rephrasing the conjecture into "the chromatic number of Kneser's graph $KG_{n,k}$ is $k + 2$ " [17]. Many more topological and combinatorial proofs followed the interest of which extends beyond the original conjecture [21]. Although Kneser's conjecture is concerned with the chromatic numbers of certain graphs (Kneser graphs), the proof methods that are known provide lower bounds for the chromatic number of any graph [18]. Thus, this initiated the application of topological tools in studying graph theory problems and more particularly in graph coloring problems [10].

The interest to provide boundaries for the chromatic number $\chi(G)$ of an arbitrary graph G through the study of different simplicial complexes associated to G , which is found in algebraic topology bibliography, drove the motivation for defining the linear coloring on the graph G and studying the relation between the chromatic number $\chi(G)$ and the linear chromatic number $\lambda(\overline{G})$. We show that for any graph G , $\lambda(\overline{G})$ is an upper bound for $\chi(G)$. The interest of this result lies on the fact that we present a linear coloring algorithm that can be applied to any graph G and provides an upper bound $\lambda(\overline{G})$ for the chromatic number of the graph G , i.e. $\chi(G) \leq \lambda(\overline{G})$; in particular, it provides a proper vertex coloring of G using $\lambda(\overline{G})$ colors. Additionally, recall that a known lower bound for the chromatic number of any graph G is the clique number $\omega(G)$ of G , i.e. $\chi(G) \geq \omega(G)$. Motivated by the definition of perfect graphs, for which $\chi(G_A) = \omega(G_A)$ holds $\forall A \subseteq V(G)$, it was interesting to study those graphs for which the equality $\chi(G) = \lambda(\overline{G})$ holds, and even more those graphs for which this equality holds for every induced subgraph. The outcome of this study was the definition of a new class of perfect graphs, namely co-linear graphs, and, furthermore, the study of the classes of co-linear graphs and of their complement class, namely linear graphs.

Our Results. In this paper, we first introduce the linear coloring of a graph G and study the relation between the linear coloring of \overline{G} and the proper vertex coloring of G . We prove that, for any graph G , a linear coloring of \overline{G} is a proper vertex coloring of G and, thus, $\lambda(\overline{G})$ is an upper bound for $\chi(G)$, i.e. $\chi(G) \leq \lambda(\overline{G})$. We present a linear coloring algorithm that can be applied to any graph G . Motivated by these results and the Perfect Graph Theorem [14], we study those graphs for which the equality $\chi(G) = \lambda(\overline{G})$ holds for every induce subgraph and define a new class of perfect graphs, namely co-linear graphs; we also study their complement class, namely linear graphs. A graph G is a *co-linear graph* if and only if its chromatic number $\chi(G)$ equals to the linear chromatic number $\lambda(\overline{G})$ of its complement graph \overline{G} , and the equality holds for every induced subgraph of G , i.e. $\chi(G_A) = \lambda(\overline{G}_A)$, $\forall A \subseteq V(G)$; a graph G is a *linear graph* if it is the complement of a co-linear graph. We show that the class of co-linear graphs is a superclass of the class of threshold graphs, a subclass of the class of co-chordal graphs and is distinguished from the class of split graphs. Additionally, we give some structural and recognition properties for the classes of linear and co-linear graphs. We study the structure of the forbidden induced subgraphs of the class of linear graphs, and show that any P_6 -free chordal graph, which is not a linear graph, properly contains a k -sun as an induced subgraph. Therefore, we infer that the subclass of chordal graphs, namely linear graphs, is a superclass of the class of P_6 -free strongly chordal graphs.

Basic Definitions. Some basic graph theory definitions follow. We consider finite undirected and directed graphs with no loops or multiple edges. Let G be such a graph; then, $V(G)$ and $E(G)$ denote the set of vertices and of edges of G , respectively. An edge is a pair of distinct vertices $x, y \in V(G)$, and is denoted by xy if G is an undirected graph and by \overrightarrow{xy} if G is a directed graph. For a set $A \subseteq V(G)$ of vertices of the graph G , the subgraph of G induced by A is denoted by G_A . Additionally, the cardinality of a set A is denoted by $|A|$. For a given vertex ordering (v_1, v_2, \dots, v_n) of a graph G , the subgraph of G induced by the set of vertices $\{v_i, v_{i+1}, \dots, v_n\}$ is denoted by G_i . The set $N(v) = \{u \in V(G) : (u, v) \in E(G)\}$ is called the *open neighborhood* of the vertex $v \in V(G)$ in G , sometimes denoted by $N_G(v)$ for clarity reasons. The set $N[v] = N(v) \cup \{v\}$ is called the *closed neighborhood* of the vertex $v \in V(G)$ in G . In a graph G , the length of a path is the number of edges in the path. The *distance* $d(v, u)$ from vertex v to vertex u is the minimum length of a path from v to u ; $d(v, u) = \infty$ if there is no path from v to u .

The greatest integer r for which a graph G contains an independent set of size r is called the *independence number* or otherwise the *stability number* of G and is denoted by $\alpha(G)$. The cardinality of the vertex set of the maximum clique in G is called the *clique number* of G and is denoted by $\omega(G)$. A *proper vertex coloring* of a graph G is a coloring of its vertices such that no two adjacent vertices are assigned the same color. The *chromatic number* $\chi(G)$ of G is the least integer k for which G admits a proper vertex coloring with k colors. For the numbers $\omega(G)$ and $\chi(G)$ of an arbitrary graph G the inequality $\omega(G) \leq \chi(G)$ holds. In particular, G is a *perfect graph* if the equality $\omega(G_A) = \chi(G_A)$ holds $\forall A \subseteq V(G)$. For more details on basic definitions in graph theory refer to [5, 14].

Next, definitions of some graph classes mentioned throughout the paper follow. A graph is called a *chordal graph* if it does not contain an induced subgraph isomorphic to a chordless cycle of four or more vertices. A graph is called a *co-chordal graph* if it is the complement of a chordal graph [14]. A hole is a chordless cycle C_n if $n \geq 5$; the complement of a hole is an antihole. A graph G is a *split graph* if there is a partition of the vertex set $V(G) = K + I$, where K induces a clique in G and I induces an independent set. Split graphs are characterized as $(2K_2, C_4, C_5)$ -free. *Threshold graphs* are defined as those graphs where stable subsets of their vertex sets can be distinguished by using a single linear inequality. Threshold graphs were introduced by Chvátal and Hammer [8] and characterized as $(2K_2, P_4, C_4)$ -free. *Quasi-threshold graphs* are characterized as the (P_4, C_4) -free graphs and are also known in the literature as trivially perfect graphs [14, 20]. A graph is *strongly chordal* if it admits a strong perfect elimination ordering. Strongly chordal graphs were introduced by Farber in [11] and are characterized completely as those chordal graphs which contain no k -sun as an induced subgraph. For more details on basic definitions in graph theory refer to [5, 14].

2 Linear Coloring on Graphs

In this section we define the linear coloring of a graph G , we prove some properties of the linear coloring of G , and present a simple algorithm for linear coloring that can be applied to any graph G . It is worth noting that similar properties of linear coloring of the neighborhood complex $\mathcal{N}(G)$ have been proved by Civan and Yalçın [9].

Definition 2.1. Let G be a graph and let $v \in V(G)$. The *clique set* of a vertex v is the set of all maximal cliques of G containing v and is denoted by $\mathcal{C}_G(v)$.

Definition 2.2. Let G be a graph. A surjective map $\kappa : V(G) \rightarrow [k]$ is called a *k -linear coloring* of G if the collection $\{\mathcal{C}_G(v) : \kappa(v) = i\}$ is linearly ordered by inclusion for all $i \in [k]$, where $\mathcal{C}_G(v)$ is the clique set of v , or, equivalently, for two vertices $v, u \in V(G)$, if $\kappa(v) = \kappa(u)$ then either $\mathcal{C}_G(v) \subseteq \mathcal{C}_G(u)$ or $\mathcal{C}_G(v) \supseteq \mathcal{C}_G(u)$. The least integer k for which G is k -linear colorable is called the *linear chromatic number* of G and is denoted by $\lambda(G)$.

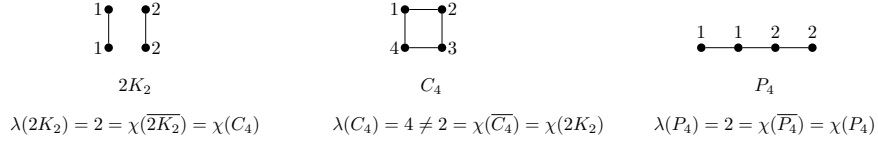


Figure 1: Illustrating a linear coloring of the graphs $2K_2$, C_4 and P_4 with the least possible colors.

2.1 Properties

Next, we study the linear coloring on graphs and its association to the proper vertex coloring. In particular, we show that for any graph G the linear chromatic number of \overline{G} is an upper bound for $\chi(G)$.

Proposition 2.1. *Let G be a graph. If $\kappa : V(G) \rightarrow [k]$ is a k -linear coloring of \overline{G} , then κ is a coloring of the graph G .*

Proof. Let G be a graph and let $\kappa : V(G) \rightarrow [k]$ be a k -linear coloring of \overline{G} . From Definition 2.2, we have that for any two vertices $v, u \in V(G)$, if $\kappa(v) = \kappa(u)$ then either $\mathcal{C}_{\overline{G}}(v) \subseteq \mathcal{C}_{\overline{G}}(u)$ or $\mathcal{C}_{\overline{G}}(v) \supseteq \mathcal{C}_{\overline{G}}(u)$ holds. Without loss of generality, assume that $\mathcal{C}_{\overline{G}}(v) \subseteq \mathcal{C}_{\overline{G}}(u)$ holds. Consider a maximal clique $C \in \mathcal{C}_{\overline{G}}(v)$. Since, $\mathcal{C}_{\overline{G}}(v) \subseteq \mathcal{C}_{\overline{G}}(u)$, then $C \in \mathcal{C}_{\overline{G}}(u)$. Thus, both $u, v \in C$ and therefore $uv \in E(\overline{G})$ and $uv \notin E(G)$. Hence, any two vertices assigned the same color in a k -linear coloring of \overline{G} are not neighbors in G . Concluding, any k -linear coloring of \overline{G} is a coloring of G . ■

It is therefore straightforward to conclude the following.

Corollary 2.1. *For any graph G , $\lambda(\overline{G}) \geq \chi(G)$.*

In Figure 1 we depict a linear coloring of the well known graphs $2K_2$, C_4 and P_4 , using the least possible colors, and show the relation between the chromatic number $\chi(G)$ of each graph $G \in \{2K_2, C_4, P_4\}$ and the linear chromatic number $\lambda(\overline{G})$.

Proposition 2.2. *Let G be a graph. A coloring $\kappa : V(G) \rightarrow [k]$ of G is a k -linear coloring of \overline{G} if and only if either $N_G(u) \subseteq N_G(v)$ or $N_G(u) \supseteq N_G(v)$ holds in G , for every $u, v \in V(G)$ with $\kappa(u) = \kappa(v)$.*

Proof. Let G be a graph and let $\kappa : V(G) \rightarrow [k]$ be a coloring of G . Assume that κ is a k -linear coloring of \overline{G} . We will show that either $N_G(u) \subseteq N_G(v)$ or $N_G(u) \supseteq N_G(v)$ holds in G for every $u, v \in V(G)$ with $\kappa(u) = \kappa(v)$. Consider two vertices $v, u \in V(G)$, such that $\kappa(u) = \kappa(v)$. Since κ is a linear coloring of \overline{G} then, from Definition 2.2, either $\mathcal{C}_{\overline{G}}(u) \subseteq \mathcal{C}_{\overline{G}}(v)$ or $\mathcal{C}_{\overline{G}}(u) \supseteq \mathcal{C}_{\overline{G}}(v)$ holds. Without loss of generality, assume that $\mathcal{C}_{\overline{G}}(u) \subseteq \mathcal{C}_{\overline{G}}(v)$. We will show that $N_G(u) \supseteq N_G(v)$ holds in G . Assume the contrary. Thus, a vertex $z \in V(G)$ exists, such that $z \in N_G(v)$ and $z \notin N_G(u)$ and, thus, $zu \in E(\overline{G})$ and $zv \notin E(\overline{G})$. Now consider a maximal clique C in \overline{G} which contains z and u . Since $zv \notin E(\overline{G})$ then $v \notin C$. Thus, there exists a maximal clique C in \overline{G} such that $C \in \mathcal{C}_{\overline{G}}(u)$ and $C \notin \mathcal{C}_{\overline{G}}(v)$, which is a contrast to our assumption that $\mathcal{C}_{\overline{G}}(u) \subseteq \mathcal{C}_{\overline{G}}(v)$. Therefore, $N_G(u) \supseteq N_G(v)$ holds in G .

Let G be a graph and let $\kappa : V(G) \rightarrow [k]$ be a coloring of G . Assume now that either $N_G(u) \subseteq N_G(v)$ or $N_G(u) \supseteq N_G(v)$ holds in G , for every $u, v \in V(G)$ with $\kappa(u) = \kappa(v)$. We will show that the coloring κ of G is a k -linear coloring of \overline{G} . Without loss of generality, assume that $N_G(u) \supseteq N_G(v)$ holds in G . We will show that $\mathcal{C}_{\overline{G}}(u) \subseteq \mathcal{C}_{\overline{G}}(v)$. Assume the opposite. Thus, a maximal clique C exists in \overline{G} , such that $C \in \mathcal{C}_{\overline{G}}(u)$ and $C \notin \mathcal{C}_{\overline{G}}(v)$. Now consider a vertex $z \in V(G)$ ($z \neq u$ and $z \neq v$), such that $z \in C$ and $zv \notin E(\overline{G})$. Such a vertex exists since C is maximal in \overline{G} and $C \notin \mathcal{C}_{\overline{G}}(v)$. Thus, $zv \notin E(\overline{G})$ and $zu \in E(\overline{G})$. Hence, $zv \in E(G)$ and $zu \notin E(G)$, which is a contrast to our assumption that $N_G(u) \supseteq N_G(v)$. ■

Taking into consideration Definition 2.2 and Proposition 2.2, we show the following.

Corollary 2.2. *Let G be a graph and let $\kappa : V(G) \rightarrow [k]$ be a k -linear coloring of \overline{G} . For every pair of vertices $u, v \in V(G)$ for which $\kappa(u) = \kappa(v)$, the following statements are equivalent:*

- (i) $\mathcal{C}_{\overline{G}}(u) \subseteq \mathcal{C}_{\overline{G}}(v)$ or $\mathcal{C}_{\overline{G}}(u) \supseteq \mathcal{C}_{\overline{G}}(v)$
- (ii) $N_G(v) \subseteq N_G(u)$ or $N_G(v) \supseteq N_G(u)$
- (iii) $N_{\overline{G}}[u] \subseteq N_{\overline{G}}[v]$ or $N_{\overline{G}}[u] \supseteq N_{\overline{G}}[v]$.

Proof. From Definition 2.2 and Proposition 2.2, it is easy to see that (i) \Leftrightarrow (ii) holds. What is left to show is (ii) \Leftrightarrow (iii), which is straightforward from basic set theory principles; specifically, take into consideration that $N_G(u) = V(G) \setminus N_{\overline{G}}[u]$, where $N_G(u)$ denotes the open neighborhood of u in G and $N_{\overline{G}}[u]$ denotes the closed neighborhood of u in \overline{G} . ■

Observation 2.1. It is easy to see that using Corollary 2.2, the definition of a linear coloring of a graph G can be restated as follows: A coloring $\kappa : V(G) \rightarrow [k]$ is a k -linear coloring of G if the collection $\{N_G[v] : \kappa(v) = i\}$ is linearly ordered by inclusion for all $i \in [k]$. Equivalently, for two vertices $v, u \in V(G)$, if $\kappa(v) = \kappa(u)$ then either $N_G[v] \subseteq N_G[u]$ or $N_G[v] \supseteq N_G[u]$.

2.2 A Linear Coloring Algorithm

In this section we present a polynomial time algorithm for linear coloring which can be applied to any graph G , and provides an upper bound for $\chi(G)$. Although we have introduced linear coloring through Definition 2.2, in our algorithm we exploit the property stated in Observation 2.1, since the problem of finding all maximal cliques of a graph G is not polynomially solvable on general graphs. Before describing our algorithm, we first construct a directed acyclic graph (DAG) D_G of a graph G , which we call *DAG associated to the graph G* , and we use it in the proposed algorithm.

The DAG D_G associated to the graph G . Let G be a graph. We first compute the closed neighborhood $N_G[v]$ of each vertex v of G , and then, we construct the following directed acyclic graph D , which depicts all inclusion relations among the vertices' closed neighborhoods: $V(D) = V(G)$ and $E(D) = \{\overrightarrow{xy} : x, y \in V(D) \text{ and } N_G[x] \subseteq N_G[y]\}$, where \overrightarrow{xy} is a directed edge from x to y . In the case where the equality $N_G[x] = N_G[y]$ holds, we choose to add one of the two edges so that the resulting graph D is acyclic (for example, we can use the labelling of the vertices, and if $x < y$ then we add \overrightarrow{xy}). It is easy to see that D is a transitive directed acyclic graph. Indeed, by definition D is constructed on a partially ordered set of elements $(V(D), \leq)$, such that for some $x, y \in V(D)$, $x \leq y \Leftrightarrow N_G[x] \subseteq N_G[y]$.

For reasons of simplicity, we consider the vertices of D located in levels. In the first level we consider the vertices with indegree equal to zero. For every vertex y belonging to level ℓ there exists at least one vertex x in level $\ell - 1$ such that \overrightarrow{xy} . For every edge \overrightarrow{xy} , if x belongs to level i and y belongs to level j , then $i < j$. For example, in the case where the equality $N_G[x] = N_G[y]$ holds, and vertices x and y are already located in levels i and j respectively, such that $i < j$, then we choose to add the edge \overrightarrow{xy} .

The algorithm for linear coloring. Given a graph G , the proposed algorithm computes a linear coloring and the linear chromatic number of G . The algorithm works as follows:

- (i) **compute** the closed neighborhood set of every vertex of G , and, then, find the inclusion relations among the neighborhood sets and construct the DAG D_G associated to the graph G .
- (ii) **find** a minimum path cover $\mathcal{P}(D_G)$, and its size $\rho(D_G)$, of the transitive DAG D_G (e.g. see [4]).
- (iii) **assign** one color $\kappa(v)$ to each vertex $v \in V(D_G)$, such that vertices belonging to the same path of $\mathcal{P}(D_G)$ are assigned the same color and vertices of different paths are assigned different colors; this is a surjective map $\kappa : V(D_G) \rightarrow [\rho(D_G)]$.

- (iv) **return** the value $\kappa(v)$ for each vertex $v \in V(D_G)$ and the size $\rho(D_G)$ of the minimum path cover of D_G ; κ is a linear coloring of G and $\rho(D_G)$ equals the linear chromatic number $\lambda(G)$ of G .

Correctness of the algorithm. Let G be a graph and let D_G be the DAG associated to the graph G . The computation of a minimum path cover in a transitive DAG D is known to be polynomially solvable; the problem is equivalent to the maximum matching problem in a bipartite graph formed from D [4]. Consider the value $\kappa(v)$ for each vertex $v \in V(D_G)$ returned by the algorithm and the size $\rho(D_G)$ of a minimum path cover of D_G . We show that the surjective map $\kappa : V(D_G) \rightarrow [\rho(D_G)]$ is a linear coloring of the vertices of G , and prove that the size $\rho(D_G)$ of the minimum path cover $\mathcal{P}(D_G)$ of the DAG D_G is equal to the linear chromatic number $\lambda(G)$ of the graph G .

Proposition 2.3. *Let G be a graph and let D_G be the DAG associated to the graph G . A path cover of D_G gives a linear coloring of the graph G by assigning a particular color to all vertices of each path. Moreover, the size $\rho(D_G)$ of the minimum path cover $\mathcal{P}(D_G)$ of the graph D_G equals to the linear chromatic number $\lambda(G)$ of the graph G .*

Proof. Let G be a graph, D_G be the DAG associated to G , and let $\mathcal{P}(D_G)$ be a minimum path cover of D_G . The size $\rho(D_G)$ of the DAG D_G , equals to the minimum number of directed paths in D_G needed to cover the vertices of D_G and, thus, the vertices of G . Now, consider a coloring $\kappa : V(D_G) \rightarrow [k]$ of the vertices of D_G , such that vertices belonging to the same path are assigned the same color and vertices of different paths are assigned different colors. Therefore, we have $\rho(D_G)$ colors and $\rho(D_G)$ sets of vertices, one for each color. For every set of vertices belonging to the same path, their corresponding closed neighborhood sets can be linearly ordered by inclusion. Indeed, consider a path in D_G with vertices $\{v_1, v_2, \dots, v_m\}$ and edges $\overrightarrow{v_i v_{i+1}}$ for $i \in \{1, 2, \dots, m\}$. From the construction of D_G , it holds that $\forall i, j \in \{1, 2, \dots, m\}, \overrightarrow{v_i v_j} \in E(D_G) \Leftrightarrow N_G[v_i] \subseteq N_G[v_j]$. In other words, the corresponding neighborhood sets of the vertices belonging to a path in D_G are linearly ordered by inclusion. Thus, the coloring κ of the vertices of D_G gives a linear coloring of G . This linear coloring κ is optimal, uses $k = \rho(D_G)$ colors, and gives the linear chromatic number $\lambda(G)$ of the graph G . Indeed, suppose that there exists a different linear coloring $\kappa' : V(D_G) \rightarrow [k']$ of G using k' colors, such that $k' < k$. For every color given in κ' , consider a set consisted of the vertices assigned that color. It is true that for the vertices belonging to the same set, their neighborhood sets are linearly ordered by inclusion. Therefore, these vertices can belong to the same path in D_G . Thus, each set of vertices in G corresponds to a path in D_G and, additionally, all vertices of G (and therefore of D_G) are covered. This is a path cover of D_G of size $\rho'(D_G) = k' < k = \rho(D_G)$, which is a contradiction since $\mathcal{P}(D_G)$ is a minimum path cover of D_G . Therefore, we conclude that the linear coloring $\kappa : V(D_G) \rightarrow [\rho(D_G)]$ is optimal, and hence, $\rho(D_G) = \lambda(G)$. ■

3 Co-linear Graphs

In Section 2 we showed that for any graph G , the linear chromatic number $\lambda(\overline{G})$ of \overline{G} is an upper bound for the chromatic number $\chi(G)$ of G , i.e. $\chi(G) \leq \lambda(\overline{G})$. Recall that a known lower bound for the chromatic number of G is the clique number $\omega(G)$ of G , i.e. $\chi(G) \geq \omega(G)$. Motivated by the Perfect Graph Theorem [14], in this section we exploit our results on linear coloring and we study those graphs for which the equality $\chi(G) = \lambda(\overline{G})$ holds for every induce subgraph. The outcome of this study was the definition of a new class of perfect graphs, namely co-linear graphs. We also prove structural properties for its members.

Definition 3.1. A graph G is called *co-linear* if and only if $\chi(G_A) = \lambda(\overline{G_A})$, $\forall A \subseteq V(G)$; a graph G is called *linear* if \overline{G} is a co-linear graph.

Next, we show that co-linear graphs are perfect; actually, we show that they form a subclass of the class of co-chordal graphs, a superclass of the class of threshold graphs and they are distinguished from the class of split graphs. We first give some definitions and show some interesting results.

Definition 3.2. The edge uv of a graph G is called *actual* if neither $N_G[u] \subseteq N_G[v]$ nor $N_G[u] \supseteq N_G[v]$. The set of all actual edges of G will be denoted by $E_\alpha(G)$.

Definition 3.3. A graph G is called *quasi-threshold* if it has no induced subgraph isomorphic to a C_4 or a P_4 or, equivalently, if it contains no actual edges.

More details on actual edges and characterizations of quasi-threshold graphs through a classification of their edges can be found in [20]. The following result directly follows from Definition 3.2 and Corollary 2.2.

Proposition 3.1. Let $\kappa : V(G) \rightarrow [k]$ be a k -linear coloring of the graph G . If the edge $uv \in E(G)$ is an actual edge of G , then $\kappa(u) \neq \kappa(v)$.

Based on Definitions 3.1 and 3.2, and Proposition 3.1, we prove the following result.

Proposition 3.2. Let G be a graph and let F be the graph such that $V(F) = V(G)$ and $E(F) = E(G) \cup E_\alpha(\overline{G})$. The graph G is a co-linear graph if and only if $\chi(G_A) = \omega(F_A)$, $\forall A \subseteq V(G)$.

Proof. Let G be a graph and let F be a graph such that $V(F) = V(G)$ and $E(F) = E(G) \cup E_\alpha(\overline{G})$, where $E_\alpha(\overline{G})$ is the set of all actual edges of \overline{G} . From Definition 3.1, G is a co-linear graph if and only if $\chi(G_A) = \lambda(\overline{G}_A)$, $\forall A \subseteq V(G)$. It suffices to show that $\lambda(\overline{G}_A) = \omega(F_A)$, $\forall A \subseteq V(G)$. From Corollary 2.2, it is easy to see that two vertices which are not connected by an edge in \overline{G}_A belong necessarily to different cliques, and thus, they cannot receive the same color in a linear coloring of \overline{G}_A . In other words, the vertices which are connected by an edge in G_A cannot take the same color in a linear coloring of \overline{G}_A . Moreover, from Proposition 3.1 vertices which are endpoints of actual edges in \overline{G}_A cannot take the same color in a linear coloring of \overline{G}_A .

Next, we construct the graph F_A with vertex set $V(F_A) = V(G_A)$ and edge set $E(F_A) = E(G_A) \cup E_\alpha(\overline{G}_A)$, where $E_\alpha(\overline{G}_A)$ is the set of all actual edges of \overline{G}_A . Every two vertices in F_A , which have to take a different color in a linear coloring of \overline{G}_A are connected by an edge. Thus, the size of the maximum clique in F_A equals to the size of the maximum set of vertices which pairwise must take a different color in \overline{G}_A , i.e. $\omega(F_A) = \lambda(\overline{G}_A)$ holds for all $A \subseteq V(G)$. Concluding, G is a co-linear graph if and only if $\chi(G_A) = \omega(F_A)$, $\forall A \subseteq V(G)$. ■

Taking into consideration Proposition 3.2 and the structure of the edge set $E(F) = E(G) \cup E_\alpha(\overline{G})$ of the graph F , it is easy to see that $E(F) = E(G)$ if \overline{G} has no actual edges. Actually, this will be true for all induced subgraphs, since if G is a quasi-threshold graph then G_A is also a quasi-threshold graph for all $A \subseteq V(G)$. Thus, $\chi(G_A) = \omega(F_A)$, $\forall A \subseteq V(G)$. Therefore, the following result holds.

Corollary 3.1. Let G be a graph. If \overline{G} is quasi-threshold, then G is a co-linear graph.

From Corollary 3.1 we obtain a more interesting result.

Proposition 3.3 Any threshold graph is a co-linear graph.

Proof. Let G be a threshold graph. It has been proved that an undirected graph G is a threshold graph if and only if G and its complement \overline{G} are quasi-threshold graphs [20]. From Corollary 3.1, if \overline{G} is quasi-threshold then G is a co-linear graph. Concluding, if G is threshold, then \overline{G} is quasi-threshold and thus G is a co-linear graph. ■

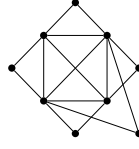


Figure 2: A graph G which is a split graph but not co-linear, since $\chi(G) = 4$ and $\lambda(\overline{G}) = 5$.



Figure 3: Illustrating the graph \overline{P}_6 which is not a co-linear graph, since $\chi(\overline{P}_6) \neq \lambda(P_6)$.

However, not any co-linear graph is a threshold graph. Indeed, Chvátal and Hammer [8] showed that threshold graphs are $(2K_2, P_4, C_4)$ -free, and, thus, the graphs P_4 and C_4 are co-linear graphs but not threshold graphs (see Figure 1). We note that the proof that any threshold graph G is a co-linear graph can be also obtained by showing that any coloring of a threshold graph G is a linear coloring of \overline{G} by using Proposition 2.2, Corollary 2.1 and the property that $N(u) \subseteq N[v]$ or $N(v) \subseteq N[u]$ for any two vertices u, v of G . However, Proposition 3.2 and Corollary 3.1 actually give us a stronger result since the class of quasi-threshold graphs is a superclass of the class of threshold graphs.

The following result is even more interesting, since it places the class of co-linear graphs into the map of perfect graphs as a subclass of co-chordal graphs.

Proposition 3.4. *Any co-linear graph is a co-chordal graph.*

Proof. Let G be a co-linear graph. It has been showed that a co-chordal graph is $(2K_2, \text{antihole})$ -free [14]. To show that any co-linear graph G is a co-chordal graph we will show that if G has a $2K_2$ or an *antihole* as induced subgraph, then G is not a co-linear graph. Since by definition a graph G is co-linear if and only if the equality $\chi(G_A) = \lambda(\overline{G}_A)$ holds for every induced subgraph G_A of G , it suffices to show that the graphs $2K_2$ and *antihole* are not co-linear graphs.

The graph $2K_2$ is not a co-linear graph, since $\chi(2K_2) = 2 \neq 4 = \lambda(C_4)$; see Figure 1. Now, consider the graph $G = \overline{C}_n$ which is an antihole of size $n \geq 5$. We will show that $\chi(G) \neq \lambda(\overline{G})$. It follows that $\lambda(\overline{G}) = \lambda(C_n) = n \geq 5$, i.e. if the graph $\overline{G} = C_n$ is to be colored linearly, every vertex has to take a different color. Indeed, assume that a linear coloring $\kappa : V(G) \rightarrow [k]$ of $\overline{G} = C_n$ exists such that for some $u_i, u_j \in V(G)$, $i \neq j$, $1 \leq i, j \leq n$, $\kappa(u_i) = \kappa(u_j)$. Since u_i, u_j are vertices of a hole, their neighborhoods in \overline{G} are $N[u_i] = \{u_{i-1}, u_i, u_{i+1}\}$ and $N[u_j] = \{u_{j-1}, u_j, u_{j+1}\}$, $2 \leq i, j \leq n-1$. For $i = 1$ or $i = n$, $N[u_1] = \{u_n, u_1, u_2\}$ and $N[u_n] = \{u_{n-1}, u_n, u_1\}$. Since $\kappa(u_i) = \kappa(u_j)$, from Corollary 2.2 we obtain that one of the inclusion relations $N[u_i] \subseteq N[u_j]$ or $N[u_i] \supseteq N[u_j]$ must hold in \overline{G} . Obviously this is possible if and only if $i = j$, for $n \geq 5$; this is a contradiction to the assumption that $i \neq j$. Thus, no two vertices in a hole take the same color in a linear coloring. Therefore, $\lambda(\overline{G}) = n$. It suffices to show that $\chi(G) < n$. It is easy to see that for the antihole \overline{C}_n , $\deg(u) = n-3$, for every vertex $u \in V(G)$. Brook's theorem [6] states that for an arbitrary graph G and for all $u \in V(G)$, $\chi(G) \leq \max\{d(u) + 1\} = (n-3) + 1 = n-2$. Therefore, $\chi(G) \leq n-2 < n = \lambda(\overline{G})$. Thus the antihole \overline{C}_n is not a co-linear graph.

We have showed that the graphs $2K_2$ and *antihole* are not co-linear graphs. It follows that any co-linear graph is $(2K_2, \text{antihole})$ -free and, thus, any co-linear graph is a co-chordal graph. ■

Although any co-linear graph is co-chordal, the reverse is not always true. For example, the graph G in Figure 2 is a co-chordal graph but not a co-linear graph. Indeed, $\chi(G) = 4$ and $\lambda(\overline{G}) = 5$. It is easy to see that this graph is also a split graph. Moreover, the class of split graphs is distinguished from the class of co-linear graphs since the graph C_4 is a co-linear graph but not a split graph, and the graph G in Figure 2 is a split graph but not a co-linear graph. However, the two classes are not disjoint; an example is the graph C_3 . Recall that a graph G is a *split graph* if there is a partition of the vertex set $V(G) = K + I$, where K induces a clique in G and I induces an independent set; split graphs are characterized as $(2K_2, C_4, C_5)$ -free graphs.

We have proved that co-linear graphs are $(2K_2, \text{antihole})$ -free. Note that, since $\overline{C_5} = C_5$ and also the chordless cycle C_n is $2K_2$ -free for $n \geq 6$, it is easy to see that co-linear graphs are *hole*-free. In addition, $\overline{P_6}$ is another forbidden induced subgraph for co-linear graphs (see Figure 3). Thus, we obtain the following result.

Proposition 3.5. *If G is a co-linear graph, then G is $(2K_2, \text{antihole}, \overline{P_6})$ -free.*

The forbidden graphs $2K_2$, *antihole*, and $\overline{P_6}$ are not enough to characterize completely the class of co-linear graphs, since split graphs do not contain any of these graphs as an induced subgraph. Thus, split graphs which are not co-linear graphs cannot be characterized by these forbidden induced subgraphs; see Figure 2.

4 Linear Graphs

In this section we study the complement class of co-linear graphs, namely linear graphs, in terms of forbidden induced subgraphs, and we derive inclusion relations between the class of linear graphs and other classes of perfect graphs.

4.1 Properties

We first provide a characterization of linear graphs by means of linear coloring on graphs. Since co-linear graphs are perfect, it follows that if G is a co-linear graph $\chi(G_A) = \omega(G_A) = \alpha(\overline{G_A})$, $\forall A \subseteq V(G)$. Therefore, the following characterization of linear graphs holds.

Proposition 4.1. *A graph G is linear if and only if $\alpha(G_A) = \lambda(G_A)$, $\forall A \subseteq V(G)$.*

From Corollary 2.1 and Proposition 4.1 we obtain the following characterization for linear graphs.

Proposition 4.2. *Linear graphs are those graphs G for which the linear chromatic number achieves its theoretical lower bound in every induced subgraph of G .*

Directly from Corollary 3.1 we can obtain the following result: any quasi-threshold graph is a linear graph. From Propositions 3.5 and 4.1 we obtain that linear graphs are (C_4, hole, P_6) -free. Therefore, the following result holds.

Proposition 4.3. *Any linear graph is a chordal graph.*

Although any linear graph is chordal, the reverse is not always true, i.e. not any chordal graph is a linear graph. For example, the complement \overline{G} of the graph illustrated in Figure 2 is a chordal graph but not a linear graph. Indeed, $\alpha(\overline{G}) = 4$ and $\lambda(\overline{G}) = 5$. It is easy to see that this graph is also a split graph. Moreover, the class of split graphs is distinguished from the class of linear graphs since the graph $2K_2$ is a linear graph but not a split graph, and the graph \overline{G} of Figure 2 is a split graph but not a linear graph. However, the two classes are not disjoint; an example is the graph C_3 .

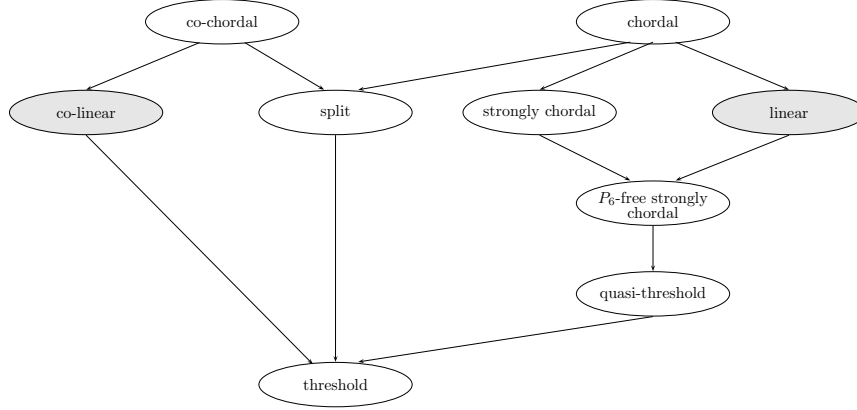


Figure 4: Illustrating the inclusion relations among the classes of linear graphs, co-linear graphs, and other classes of perfect graphs.

Another known subclass of the class of chordal graphs is the class of strongly chordal graphs. The following definitions and results given by Farber [11] turn up to be useful in proving some results about the structure of linear graphs. More details about strongly chordal graphs can be found in [5, 11].

Definition 4.2. (Farber [11]) A vertex ordering (v_1, v_2, \dots, v_n) is a *strong perfect elimination ordering* of a graph G iff σ is a perfect elimination ordering and also has the property that for each i, j, k and ℓ , if $i < j$, $k < \ell$, $v_k, v_\ell \in N[v_i]$, and $v_k \in N[v_j]$, then $v_\ell \in N[v_j]$. A graph is *strongly chordal* iff it admits a strong perfect elimination ordering.

Definition 4.3. (Farber [11]) Let G be a graph. A vertex v is *simple* in G if $\{N[x] : x \in N[v]\}$ is linearly ordered by inclusion.

Theorem 4.1. (Farber [11]) A graph G is strongly chordal if and only if every induced subgraph of G has a simple vertex.

Corollary 4.1. (Chang [7]) A strong perfect elimination ordering of a graph G is a vertex ordering (v_1, v_2, \dots, v_n) such that for all $i \in \{1, 2, \dots, n\}$ the vertex v_i is simple in G_i and also $N_{G_i}[v_\ell] \subseteq N_{G_i}[v_k]$ whenever $i \leq \ell \leq k$ and $v_\ell, v_k \in N_{G_i}[v_i]$.

The following characterization of strongly chordal graphs will be next used to derive properties about the structure of linear graphs. We first give the following definition.

Definition 4.1. An *incomplete k -sun* S_k ($k \geq 3$) is a chordal graph on $2k$ vertices whose vertex set can be partitioned into two sets, $U = \{u_1, u_2, \dots, u_k\}$ and $W = \{w_1, w_2, \dots, w_k\}$, so that W is an independent set, and w_i is adjacent to u_j if and only if $i = j$ or $i = j + 1 \pmod{k}$. A k -sun is an incomplete k -sun S_k in which U is a complete graph.

Proposition 4.4. (Farber [11]) A chordal graph G is strongly chordal if and only if it contains no induced k -sun.

4.2 Forbidden Subgraphs

Hereafter, we study the structure of the forbidden induced subgraphs of the class of linear graphs, and we prove that any P_6 -free chordal graph which is not a linear graph properly contains a k -sun as an induced subgraph.

We consider the class of P_6 -free chordal graphs which we have shown that it properly contains the class of linear graphs. Let \mathcal{F} be the family of all the minimal forbidden induced subgraphs of the class of linear graphs. Let F_i be a member of \mathcal{F} , which is neither a C_n ($n \geq 4$) nor a P_6 . We next prove the main result of this section: any graph F_i properly contains a k -sun ($k \geq 3$) as an induced subgraph. From Proposition 4.4 it suffices to show that any P_6 -free strongly chordal graph is a linear graph and also that the k -sun ($k \geq 3$) is a linear graph.

Let G be a P_6 -free strongly chordal graph. In order to show that G is a linear graph we will show that $\alpha(G) = \lambda(G)$ and that the equality holds for every induced subgraph of G . Let L be the set of all simple vertices of G , and S be the set of all simplicial vertices of G ; note that $L \subseteq S$ since a simple vertex is also a simplicial vertex. First, we construct a maximum independent set I and a strong perfect elimination ordering σ of G with special properties needed for our proof. Next, we assign a coloring $\kappa : V(G) \rightarrow [k]$ to the vertices of G , where $k = \alpha(G) = |I|$, and show that κ is an optimal linear coloring of G . Actually, we show that we can assign a linear coloring with $\lambda(G) = \alpha(G)$ colors to any P_6 -free strongly chordal graph, by using the constructed strong perfect elimination ordering σ of G . Finally, we show that the equality $\lambda(G_A) = \alpha(G_A)$ holds for every induced subgraph G_A of G .

Construction of I and σ . Let G be a P_6 -free strongly chordal graph, and let L be the set of all simple vertices in G . From Definition 4.2, G admits a strong perfect elimination ordering. Using a modified version of the algorithm given by Farber in [11] we construct a strong perfect elimination ordering $\sigma = (v_1, v_2, \dots, v_n)$ of the graph G having specific properties. Our algorithm also constructs the maximum independent set I of G . Since G is a chordal graph and σ is a perfect elimination ordering, we can use a known algorithm (e.g. see [14]) to compute a maximum independent set of the graph G . Throughout the algorithm, we denote by G_i the subgraph of G induced by the set of vertices $V(G) \setminus \{v_1, v_2, \dots, v_{i-1}\}$, where v_1, v_2, \dots, v_{i-1} are the vertices which have already been added to the ordering σ during the construction. Moreover, we denote by I^* the set of vertices which have not been added to σ yet and additionally do not have a neighbor already added in σ which belongs to I .

In Figure 5, we present a modified version of the algorithm given by Farber [11] for constructing a strong perfect elimination ordering σ of G . Our algorithm in each iteration of Steps 3–5 adds to the ordering σ all vertices which are simple in G_i , while Farber's algorithm selects only one simple vertex of G_i and adds it to σ . We note that L_i is the set of all the simple vertices of G_i and v_i is that vertex of L_i which is added first to the ordering σ . It is easy to see that the constructed ordering σ is a strong perfect elimination ordering of G , since every vertex which is simple in G is also simple in every induced subgraph of G . Clearly, the constructed set I is a maximum independent set of G .

From the fact that G is a P_6 -free strongly chordal graph and from the construction of I and σ we obtain the following properties.

Property 4.1. Let G be a P_6 -free strongly chordal graph and let L be the set of all simple vertices of G . For each vertex $v_x \notin L$, there exists a chordless path of length at most 4 connecting v_x to any vertex $v \in L$.

Property 4.2. Let G be a P_6 -free strongly chordal graph, L be the set of all simple vertices of G , and let I and σ be the maximum independent set and the ordering, respectively, constructed by our algorithm. Then,

- (i) if $v_i \notin L$ and $i < j$, then $v_j \notin L$;
- (ii) for each vertex $v_x \notin I$, there exists a vertex $v_i \in I$, $i < x$, such that $v_x \in N_{G_i}[v_i]$.

Next, we describe an algorithm for assigning a coloring κ to the vertices of G using exactly $\alpha(G)$ colors and, then, we show that κ is a linear coloring of G .

Input: a strongly chordal graph G ;

Output: a strong perfect elimination ordering σ of G ;

1. **set** $I = \emptyset$, $I^* = V(G)$, $\sigma = \emptyset$, $n = |V(G)|$, and $V_0 = V(G)$;
2. Let $(V_0, <_0)$ be the partial ordering on V_0 in which $v <_0 u$ if and only if $v = u$.
set $V_1 = V(G)$ and $i = 1$;
3. Let G_i be the subgraph of G induced by V_i , that is, $V_i = V(G_i)$.
construct an ordering on V_i by $v <_i u$ if $v <_{i-1} u$ or $N_i[v] \subset N_i[u]$;
set $k = i$;
4. Let L_k be the set of all the simple vertices in G_i .
while $L_k \neq \emptyset$ **do**
 - **construct** an ordering on V_i by $v <_i u$ if $v <_{i-1} u$ or $N_i[v] \subset N_i[u]$;
 - **choose** a vertex v_i which belongs to L_k and is minimal in $(V_i, <_i)$ to add to the ordering;
 - **set** $V_{i+1} = V_i \setminus \{v_i\}$ and $L_k = L_k \setminus \{v_i\}$;
 - **if** $v_i \in I^*$ **then**
 - set** $I = I \cup \{v_i\}$ and $I^* = I^* \setminus \{v_i\}$;
 - delete** all neighbors of v_i from I^* ;
 - **set** $i = i + 1$;**end-while**;
5. **if** $i = n + 1$ **then** output the ordering $\sigma = (v_1, v_2, \dots, v_n)$ of $V(G)$ and **stop**;
else go to step 3;

Figure 5: A modified version of Farber's algorithm for constructing a strong perfect elimination ordering σ and a maximum independent set I of a strongly chordal graph G .

The coloring κ of G . Let G be a P_6 -free strongly chordal graph, and let L (resp. S) be the set of all simple (resp. simplicial) vertices in G . We consider a maximum independent set I , and a strong elimination ordering σ , as constructed above. Now, in order to compute the linear chromatic number $\lambda(G)$ of G , we assign a coloring κ to the vertices of G and show that κ is a linear coloring of G . Actually, we show that we can assign a linear coloring with $\lambda(G) = \alpha(G)$ colors to any P_6 -free strongly chordal graph, by using the constructed strong perfect elimination ordering σ of G .

First, we assign a coloring $\kappa : V(G) \rightarrow [k]$, where $k = \alpha(G)$, to the vertices of G as follows:

1. Successively visit the vertices in the ordering σ from left to right, and color the first vertex $v_i \in I$ which has not been assigned a color yet, with color $\kappa(v_i)$.
2. Color all uncolored vertices $v_k \in N_{G_i}(v_i)$, with color $\kappa(v_k) = \kappa(v_i)$.
3. Repeat steps 1 and 2 until there are no uncolored vertices $v_i \in I$ in G .

Based on this process, we obtain that every vertex v_i belonging to the maximum independent set I of G is assigned a different color in step 1, and for each such vertex v_i all its uncolored neighbors to its right in the ordering σ are assigned the same color with v_i in step 2. Therefore, so far we have assigned $\alpha(G)$ colors to the vertices of G . Now, from Property 4.2(ii) it is easy to see that κ is a coloring of the vertex set $V(G)$, i.e. there is no vertex in σ which has not been assigned a color. Thus, κ is a coloring

of G using $\alpha(G)$ colors. Note that κ is not a proper vertex coloring of G . Actually, since the following lemma holds, from Proposition 2.1 it appears that κ is a proper vertex coloring of \overline{G} .

Lemma 4.1. *The coloring κ is a linear coloring of G .*

Proof. Let G be a P_6 -free strongly chordal graph, and let L (resp. S) be the set of all simple (resp. simplicial) vertices in G . We consider a maximum independent set I , a strong elimination ordering σ , and a coloring κ of G , as constructed above. Hereafter, for two vertices v_i and v_j in the ordering σ , we say that $v_i < v_j$ if the vertex v_i appears before the vertex v_j in σ .

Next, we show that κ is a linear coloring of G , that is, the collection $\{\mathcal{C}_G(v_\ell) : \kappa(v_\ell) = j\}$ is linearly ordered by inclusion for all $j \in [k]$. From Corollary 2.2, it is equivalent to show that the collection $\{N_G[v_\ell] : \kappa(v_\ell) = j\}$ is linearly ordered by inclusion for all $j \in [k]$. Each such collection contains exactly one set $N_G[v_i]$ where $v_i \in I$, and some sets $N_G[v_k]$ where v_k are neighbors of v_i in G_i and $\kappa(v_k) = \kappa(v_i)$. Thus, it suffices to show that for each vertex $v_i \in I$, the collection $\{N_G[v_k] : v_k \in N_{G_i}[v_i] \text{ and } \kappa(v_k) = \kappa(v_i)\}$ is linearly ordered by inclusion. To this end, we distinguish two cases regarding the vertices $v_i \in I$; in the first case we consider v_i to be a simplicial vertex, that is $v_i \in S$, and in the second case we consider $v_i \notin S$.

Case 1: The vertex $v_i \in I$ and $v_i \in S$. Since σ is a strong elimination ordering, each vertex $v_i \in I$ is simple in G_i and thus $\{N_{G_i}[v_k] : v_k \in N_{G_i}[v_i]\}$ is linearly ordered by inclusion. We will show that $\{N_G[v_k] : v_k \in N_{G_i}[v_i] \text{ and } \kappa(v_k) = \kappa(v_i)\}$ is linearly ordered by inclusion for all vertices $v_i \in I \cap S$. Recall that in the coloring κ of G we assign the color $\kappa(v_k) = \kappa(v_i)$ to a vertex $v_k \notin I$, if $v_i \in I$, $v_k \in N_{G_i}[v_i]$ and there exists no vertex $v_{i'} \in I$ such that $v_k \in N_{G_{i'}}[v_{i'}]$ and $v_{i'} < v_i$ in σ . By definition, if $v_i \in L$ then the collection $\{N_G[v_k] : v_k \in N_{G_i}[v_i] \text{ and } \kappa(v_k) = \kappa(v_i)\}$ is linearly ordered by inclusion. Thus, hereafter we consider vertices $v_i \in I \cap S$ and $v_i \notin L$.

Consider that the vertex v_i has a neighbor v_1 to its left in the ordering σ , i.e. $v_1 < v_i$. Since v_i is a simplicial vertex in G , its closed neighborhood forms a clique and, thus, $v_1 \in N_G[v_k]$ for all vertices $v_k \in N_{G_i}[v_i]$. Therefore, the existence of such a vertex v_1 preserves the linear order by inclusion of $\{N_{G_i}[v_k] \cup \{v_1\} : v_k \in N_{G_i}[v_i]\}$. Thus, $N_G[v_i] \subseteq N_G[v_k]$, for all vertices $v_k \in N_{G_i}[v_i]$ and $\kappa(v_k) = \kappa(v_i)$.

Now, consider that the vertex v_i has two neighbors v_k and v_j to its right in the ordering σ , such that $v_i < v_k < v_j$ and $\kappa(v_k) = \kappa(v_j) = \kappa(v_i)$; thus, $N_{G_i}[v_k] \subseteq N_{G_i}[v_j]$. In the case where the equality $N_{G_i}[v_k] = N_{G_i}[v_j]$ holds, without loss of generality, we may assume that the degree of v_k in G is less than or equal to the degree of v_j in G (note that σ is still a strong elimination ordering). Assume that $N_G[v_k] \subseteq N_G[v_j]$ does not hold. Then, there exist vertices v_2 and v_3 in G such that $v_2 \in N_G[v_k]$, $v_2 \notin N_G[v_j]$, $v_3 \in N_G[v_j]$, and $v_3 \notin N_G[v_k]$. Since $N_{G_i}[v_k] \subseteq N_{G_i}[v_j]$, it is easy to see that $v_2 < v_i$ in σ . Assume that v_2 is the first (from left to right) neighbor of v_k in σ . Since $\kappa(v_k) = \kappa(v_i)$, it follows that $v_2 \notin I$. Moreover, from Property 4.2(ii) it holds that there exists a vertex $v_4 \in I$, such that $v_4 < v_2$ and $v_2 \in N_G[v_4]$. Additionally, since $\kappa(v_k) = \kappa(v_j) = \kappa(v_i)$ it holds that $v_k, v_j \notin N_G[v_4]$. Hence, the subgraph of G induced by the vertices $\{v_4, v_2, v_k, v_j\}$ is a P_4 . Concerning now the position of the vertex v_3 in the ordering σ , we can have either $v_3 < v_i$ in the case where $N_{G_i}[v_k] = N_{G_i}[v_j]$ holds, or $v_3 > v_i$ otherwise. We will show that in both cases we are led to a contradiction to our initial assumptions; that is, either it results that G has a P_6 as an induced subgraph or that the vertices should be added to σ in an order different to the one originally assumed.

Case 1.1. $v_3 < v_i$. It is easy to see that $v_3 \notin I$, since otherwise v_j would have taken the color $\kappa(v_j) = \kappa(v_3)$ during the coloring κ of G . Thus, from Property 4.2(ii) there exists a vertex $v_5 \in I$, such that $v_5 < v_3$ and $v_3 \in N_G[v_5]$. Therefore, the vertices $\{v_4, v_2, v_k, v_j, v_3, v_5\}$ induce a P_6 in G , which is also chordless since G is chordal.

Case 1.2. $v_3 > v_i$. Since $v_i \notin L$, from Property 4.2(i) it follows that $v_3 \notin L$. Thus, from Property 4.1 we obtain that there exists a chordless path of length at most 4 connecting $v_3 \notin L$ to any vertex $v \in L$.

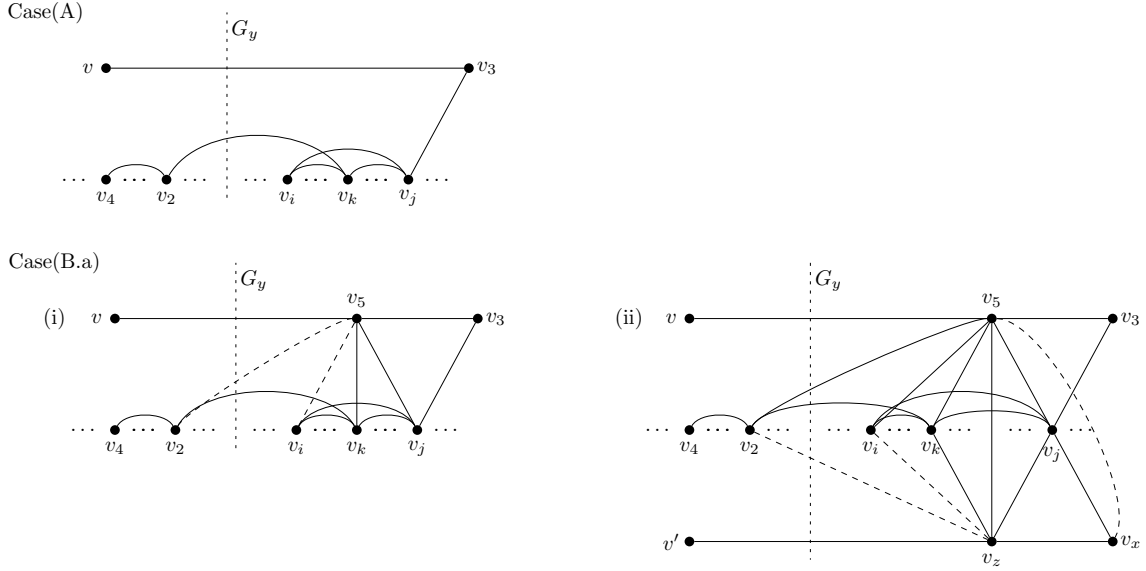


Figure 6: Illustrating Case (A) and Case (B.a)

Similarly, it easily follows that $v_4 \in L$. However, we know that in a non-trivial strongly chordal graph there exist at least two non adjacent simple vertices [11]. Thus, there exist a vertex $v \in L$, $v \neq v_4$, such that the distance $d(v, v_3)$ of v_3 from v is at most 4. Let $d_m(v_3, v) = \max\{d(v_3, v) : \forall v \in L, v \neq v_4\}$. Since $v_3 \notin L$ and G is P_6 -free, it follows that $1 \leq d_m(v, v_3) \leq 4$.

Next, we distinguish four cases regarding the maximum distance $d_m(v_3, v)$ and show that each one comes to a contradiction. In each case we have that $\{v_4, v_2, v_k, v_j, v_3\}$ is a chordless path on five vertices. We first explain what is illustrated in Figures 6 and 7. Let G_y be the induced subgraph of G , such that during the construction of σ the vertex v_i is simple in G_y , i.e. $v_i \in L_y$ and $v_y \leq v_i$. In the two figures, the vertices are placed on the horizontal dotted line in the order that appear in the ordering σ . For the vertices which are not placed on the dotted line, we are only interested about illustrating the edges among them. The vertices which are to the right of the vertical dashed line belong to the induced subgraph G_y of G . The dashed edges illustrate edges that may or may not exist in the specific case. Next, we distinguish the four cases, and show that each one of them comes to a contradiction:

Case (A): $d_m(v_3, v) = 1$.

It is easy to see that $v_j v \notin E(G)$, since otherwise v_j would have been assigned the color $\kappa(v)$ and not $\kappa(v_i)$ as assumed. Thus, in this case there exists a P_6 in G induced by the vertices $\{v_4, v_2, v_k, v_j, v_3, v\}$; since G is a chordal graph, other edges among the vertices of this path do not exist. This is a contradiction to our assumption that G is a P_6 -free graph.

Case (B): $d_m(v_3, v) = 2$.

In this case there exists a vertex v_5 such that $\{v_3, v_5, v\}$ is a chordless path from v_3 to v . It follows that there exists a P_7 induced by the vertices $\{v_4, v_2, v_k, v_j, v_3, v_5, v\}$. Having assumed that G is a P_6 -free graph, the path $\{v_4, v_2, v_k, v_j, v_3\}$ is chordless and $v_j, v_k \notin N_G[v]$, we obtain that $v_j v_5 \in E(G)$ and $v_k v_5 \in E(G)$. Next, we distinguish three cases regarding the neighborhood of the vertex v_3 in G and show that each one comes to a contradiction.

(B.a) The vertex v_3 does not have neighbors in G other than v_5 and v_j . In Case (i) we examine the cases where either $v_2 v_5 \notin E(G)$ or $v_2 v_5 \in E(G)$ and v_j does not have a neighbor v_x in G_i , such that $v_x v_k \notin E(G)$. In Case (ii) we examine the case where $v_2 v_5 \in E(G)$ and v_j has a neighbor v_x in G_i , such that $v_x v_k \notin E(G)$.

- (i) Assume that $v_2 v_5 \notin E(G)$. In this case, we can see that during the construction of σ , after the first iteration where v and v_4 are added in the ordering, the vertex v_3 becomes simple in the remaining induced subgraph of G , since $N[v_5]$ becomes a subset of $N[v_j]$. Thus, v_3 can be added to σ during the second iteration of the algorithm, along with v_2 . However, v_i will not be added to the ordering before the third iteration, since v_i is not simple before v_2 is added to σ . Thus, we conclude that v_3 will be added in σ before v_i , and more specifically that $v_3 < v_y \leq v_i$, and this is a contradiction to our assumption that $v_3 > v_i$.

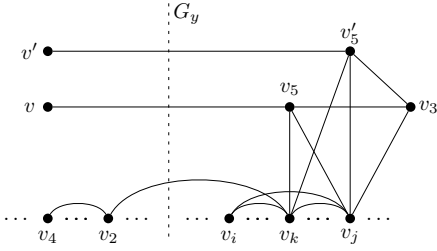
Now, assume that $v_2 v_5 \in E(G)$. We know that v_2 is simple in the subgraph G_2 of G induced by the vertices to the right of v_2 in σ . If $v_5, v_k \in N_{G_2}[v_2]$, $v_3 \in N_{G_2}[v_5]$, and $v_3 \notin N_{G_2}[v_k]$, then $N_{G_2}[v_5] \supset N_{G_2}[v_k]$. More specifically, since we have assumed that v_2 is the first (from left to right) neighbor of v_k in σ , it follows that $N_G[v_5] \supset N_G[v_k]$. We know that $N_{G_i}[v_k] \subset N_{G_i}[v_j]$, and since we have assumed that v_j does not have a neighbor v_x , such that $v_x < v_i$, it easily follows that $N_{G_i}[v_k] \subset N_{G_i}[v_j] = N_G[v_j]$. Thus, for every neighbor of v_j in G , which is also a neighbor of v_k , we obtain that it is a neighbor of v_5 as well.

Therefore, in the case where v_j does not have a neighbor v_x in G , and thus in G_i , such that $v_x v_k \notin E(G)$, it follows that $N_G[v_5]$ is a superset of $N_G[v_j]$ and, thus, the vertex v_3 is simple in G . Again we conclude that v_3 will be added to σ before v_i , and more specifically that $v_3 < v_y \leq v_i$. This is a contradiction to our assumption that $v_3 > v_i$.

- (ii) Consider now the case where $v_2 v_5 \in E(G)$ and v_j has a neighbor v_x in G , and thus in G_i , such that $v_x v_k \notin E(G)$. We will show that in this case either v_3 is simple after the first iteration, i.e. $v_x \in N[v_5]$ or v_x becomes simple after the first iteration. Since $v_x > v_i$ it follows that $v_x \notin L$. Therefore, there exists a path in G from v_x to a vertex $v' \in L$ of length $d(v_x, v')$ at most 4. Consider the case where $d(v_x, v') = 1$. If $v \equiv v'$, then $v_5 v_x \in E(G)$, since G is a chordal graph; thus, $N[v_5] \supseteq N[v_j]$ and $v_3 \in L$. It is easy to see that $v' \neq v_4$, since G is a chordal graph. Therefore, in the case where $v' v_x \in E(G)$, the graph G has a P_6 induced by the vertices $\{v_4, v_2, v_k, v_j, v_x, v'\}$. Thus, $v' v_x \notin E(G)$ and there exists a vertex v_z such that $\{v_x, v_z, v'\}$ is a chordless path from v_x to v' . Therefore, there exists a P_7 in G and, thus, $v_k, v_j \in N_G[v_z]$. Additionally, from Case(B.a)(i) we have that $v_5 \in N_G[v_z]$ (recall that if $v_2 v_5 \in E(G)$, then $N_G[v_5] \supset N_G[v_k]$).

Note that, the vertices v_x and v_z play the same role in G as the vertices v_3 and v_5 , respectively. Therefore, in the case where $v_2 v_z \notin E(G)$, the vertex v_x is simple after the first iteration and will be added to σ during the second iteration, while v_i will be added during the third. Thus, we will have $v_x < v_y < v_i$ which is a contradiction to our assumption that $v_x > v_i$. Consider now the case where $v_2 v_z \in E(G)$. Since v_2

Case(B.b)



Case(B.c)

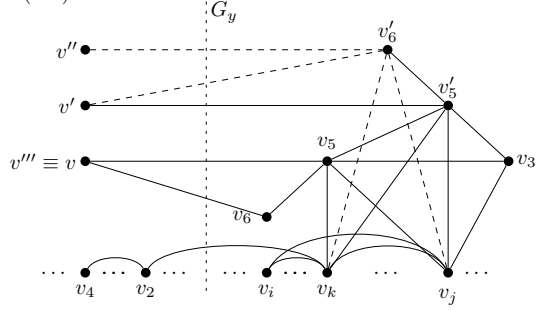


Figure 7: Illustrating Cases (B.b) and (B.c) of the proof.

is simple in the subgraph G_2 of G induced by the vertices to the right of v_2 in σ , we must have either $v_z v_3 \in E(G)$ or $v_5 v_x \in E(G)$. Without loss of generality assume that $v_5 v_x \in E(G)$. Concluding, we have shown that even in the case where v_j has a neighbor v_x in G , and thus in G_i , such that $v_x v_k \notin E(G)$, then $N_G[v_5]$ is a superset of $N_G[v_j]$, and thus $v_3 \in L$. Thus, we have again $v_3 < v_y < v_i$ which is a contradiction to our assumption that $v_3 > v_i$. The same holds even if, additionally to the other edges, $v_4 v_5 \in E(G)$.

So far, we have shown that if v_3 has the vertices v_j and v_5 as neighbors, then either $v_3 \in L$ or v_3 is simple in the second iteration, that is before v_i can be added to σ (i.e. $v_3 < v_y \leq v_i$). This is due to the fact that for any neighbor v_5 of v_3 we have shown that $N[v_5] \subseteq N[v_j]$ in the case where $v_2 v_5 \notin E(G)$, and $N[v_5] \supseteq N[v_j]$ in the case where $v_2 v_5 \in E(G)$; thus v_3 will be added to σ before v_i . Since we initially assumed that $v_3 > v_i$ in σ , i.e. that v_3 does not become simple before v_i becomes simple, we continue by examining the cases where v_3 has neighbors in G_y other than v_5 and v_j .

- (B.b) The vertex v_3 has two neighbors v_5 and v'_5 in G_y , such that $v_5 v'_5 \notin E(G)$. Since we have assumed that the maximum distance of the vertex v_3 from v in G , for any vertex $v \in L$, $v \neq v_4$, is $d_m(v_3, v) = 2$, and v_3 has no neighbor belonging to L , it follows that $v_5, v'_5 \notin L$ and there exist vertices $v, v' \in L$ such that the vertices $\{v_3, v_5, v\}$ induce a chordless path from v_3 to v and $\{v_3, v'_5, v'\}$ induce a chordless path from v_3 to v' . It is easy to see that $v \neq v'$ and $vv' \notin E(G)$ since G is a chordal graph. Therefore, from Case (B.a) we have $v_k, v_j \in N_G[v_5]$ and $v_k, v_j \in N_G[v'_5]$. However, in this case there exists a C_4 in G induced by the vertices $\{v_5, v_3, v'_5, v_k\}$, since by assumption $v_5 v'_5 \notin E(G)$ and $v_3 v_k \notin E(G)$. It easily follows that the same arguments hold for any two neighbors of v_3 in G . Concluding, the vertex v_3 cannot have two neighbors v_5 and v'_5 in G , such that $v_5 v'_5 \notin E(G)$. Thus, $v_3 \in S$.
- (B.c) The vertex v_3 has two neighbors v_5 and v'_5 (where $v_5 \neq v_j$ and $v'_5 \neq v_j$) in G_y , such that $v_5 v'_5 \in E(G)$, but neither $N_y[v_5] \subseteq N_y[v'_5]$ nor $N_y[v'_5] \subseteq N_y[v_5]$; thus, there exist vertices v_6 and v'_6 in G_y such that $v_5 v_6 \in E(G)$ and $v_5 v'_6 \notin E(G)$ and, also, $v'_5 v'_6 \in E(G)$ and $v'_5 v_6 \notin E(G)$. Since $v_3 \in S$, it follows that $v_6, v'_6 \notin N_G[v_3]$. Since $d_m(v_3, v) = 2$, there exists a vertex $v \in L$ such that $\{v_3, v_5, v\}$ is a chordless path from v_3 to v . Similarly, there exists a vertex $v' \in L$ such that $\{v_3, v'_5, v'\}$ is a chordless path from v_3 to v' . We have that $v \neq v'$, $vv'_5 \notin E(G)$ and $v'v_5 \notin E(G)$, since otherwise v and v' would not be simple in G . Additionally, $vv' \notin E(G)$, $vv'_6 \notin E(G)$, and $v'v_6 \notin E(G)$, since G is a chordal graph. Therefore, from Case (B.a) we have $v_k, v_j \in N_G[v_5]$ and $v_k, v_j \in N_G[v'_5]$. Assume that there

exist vertices $v'', v''' \in L$, such that $v_6 v''' \in E(G)$ and $v'_6 v'' \in E(G)$. It is easy to see that at least one of the equivalences $v \equiv v'''$ and $v' \equiv v''$ holds, otherwise G has a P_6 induced by the vertices $\{v''', v_6, v_5, v'_5, v'_6, v''\}$. Without loss of generality, assume that $v \equiv v'''$ holds.

Since $v \in L$, $v_5, v_6 \in N_G[v]$, $v'_5 \in N_G[v_5]$, and $v'_5 \notin N_G[v_6]$, it follows that $N_G[v_6] \subset N_G[v_5]$. In the case where $v_k, v_j \notin N_G[v_6]$ we have $v_6 \in L$ and, thus, v_6 would be added to σ in the first iteration which is a contradiction to our assumption that $v_6 \in G_y$. Assume that $v_j v_6 \in E(G)$; it follows that $v_k v_6 \in E(G)$, since otherwise G has a P_6 induced by the vertices $\{v_4, v_2, v_k, v_j, v_6, v\}$. If $v' \equiv v''$, the same arguments hold for v'_6 too and, thus, if $v_j v'_6 \in E(G)$ then $v_k v'_6 \in E(G)$. In the case where $v' \neq v''$ we have $v'_6 v_k \in E(G)$, since otherwise G has a P_6 induced by the vertices $\{v_4, v_2, v_k, v'_5, v'_6, v''\}$. Thus, in any case $v_6, v'_6 \in N_G[v_k]$, and G has a 3-sun induced by the vertices $\{v_k, v_5, v'_5, v'_6, v_6, v_3\}$. Since other edges between the vertices of the 3-sun do not exist, it follows that at least one of the vertices v_6 and v'_6 does not belong to the neighborhood of v_k and, thus, of v_j in G . Without loss of generality, let v_6 be that vertex. Thus, $v_6 \in L$ and, subsequently, v_6 will be added to σ during the first iteration. Thus, v_3 is simple and will be added to σ during the second iteration, along with v_2 , while v_i will be added to σ after the second iteration (i.e. $v_3 < v_y \leq v_i$). This is a contradiction to our assumption that $v_3 > v_i$.

Using similar arguments, we can prove that v_3 will be added to σ before v_i , even if there exist edges between v_2 and the vertices v_5, v'_5, v_6 , and v'_6 . Actually, it easily follows that $v_2 v_6 \notin E(G)$, since $v_6 v_k \notin E(G)$ and G is a chordal graph. Additionally, $v_2 v_5 \notin E(G)$, since we know that $v_5 v'_6 \notin E(G)$, $v_k v_3 \notin E(G)$ and v_2 is simple in G_2 . Therefore, whether $v_2 v'_5, v_2 v'_6 \in E(G)$ or not, it does not change the fact that v_3 becomes simple after the first iteration and, thus, v_3 is added to σ before v_i . Note, that even in the case where $v \equiv v_4$ or $v' \equiv v_4$, it similarly follows that $v'_6 \in L$ or $v_6 \in L$ respectively and, thus, v_3 becomes simple after the first iteration and is added to σ before v_i .

Case (C): $d_m(v_3, v) = 3$.

In this case there exist vertices v_5 and v_6 such that $\{v_3, v_5, v_6, v\}$ is a chordless path from v_3 to v . Since now G has a P_8 , it follows that $v_5 v_j \in E(G)$ and, additionally, some other edges must exist among the vertices v_2, v_k, v_j, v_5 , and v_6 . In any case, we will prove that either $N_G[v_5] \subseteq N_G[v_j]$ or $N_G[v_j] \subseteq N_G[v_5]$ and, thus, $v_3 \in L$. Similarly to Case (B), we distinguish three cases regarding the neighborhood of the vertex v_3 in G and show that if $v_3 \notin L$ then each one comes to a contradiction.

(C.a) The vertex v_3 does not have neighbors in G other than v_5 and v_j . Consider the case where $v_3 \notin L$ because $v_6 \notin N_G[v_j]$ and $v_k \notin N_G[v_5]$. In this case, G has a P_7 induced by the vertices $\{v_4, v_2, v_k, v_j, v_5, v_6, v\}$ which is chordless since G is a chordal graph; this is a contradiction to our assumption that G is P_6 -free. Consider, now, the case where $v_3 \notin L$ because $v_6 \notin N_G[v_j]$ and $v_i \notin N_G[v_5]$. Since G is P_6 -free it follows that $v_5 v_k \in E(G)$ and $v_6 v_k \in E(G)$. However, in this case G has a 3-sun, unless either $v_i v_6 \in E(G)$ and, thus, $v_j v_6 \in E(G)$, or $v_i v_5 \in E(G)$. In either case it follows that $v_3 \in L$.

Consider, now, the case where v_j has another neighbor v_x in G_i such that $v_x v_5 \notin E(G)$. Using similar arguments as in Case (B.a)(ii), we come to a contradiction to our assumptions. More specifically, in the case where $v_2 v_5 \in E(G)$, it is proved that $N_G[v_5] \supset N_G[v_j]$, and thus $v_3 \in L$. Similarly, in the case where $v_6 v_j \notin E(G)$, it is proved that the vertex v_x will be simple after the first iteration during the construction of σ , and thus $v_x < v_y \leq v_i$.

(C.b) The vertex v_3 has two neighbors v_5 and v'_5 in G_y , such that $v_5 v'_5 \notin E(G)$. Using the same arguments as in Case (B.b), we obtain that in this case G has a C_4 which is a contradiction to our assumptions.

(C.c) The vertex v_3 has two neighbors v_5 and v'_5 (where $v_5 \neq v_j$ and $v'_5 \neq v_j$) in G_y , such that $v_5v'_5 \in E(G)$, and neither $N_y[v_5] \subseteq N_y[v'_5]$ nor $N_y[v'_5] \subseteq N_y[v_5]$; that is, there exist vertices v_6 and v'_6 in G_y such that $v_5v_6 \in E(G)$ and $v_5v'_6 \notin E(G)$ and, also, $v'_5v'_6 \in E(G)$ and $v'_5v_6 \notin E(G)$. Similarly to Case (B.c), we can prove that this case comes to a contradiction as well. Note that, in this case $d_m(v_3, v) = 3$ and, thus, there exists a chordless path $\{v_3, v_5, v_7, v\}$ from v_3 to v . Again, at least one of $v \equiv v'''$ and $v' \equiv v''$ must hold, since otherwise G has a P_6 induced by the vertices $\{v''', v_6, v_5, v'_5, v'_6, v''\}$. Using the same arguments as in Case (B.c), we obtain that if $v \equiv v'''$ then $v_k, v_j \notin N_G[v_6]$. However, now, we must additionally have $v_6v_7 \in E(G)$, since otherwise G has a C_4 induced by the vertices $\{v, v_7, v_5, v_6\}$. Therefore, as in Case (B.c) we obtain $v_6 \in L$, which is a contradiction to our assumption that the vertex v_i appears in the ordering before the vertices v_6, v'_6, v_5 , and v'_5 .

Case (D): $d_m(v_3, v) = 4$.

In this case there exist vertices v_5, v_6 and v_7 such that $\{v_3, v_5, v_6, v_7, v\}$ is a chordless path from v_3 to v . Since now G has a P_9 , it follows that $v_5v_j \in E(G)$ and, additionally, some other edges must exist. Similarly to Cases (A) and (B), we distinguish three cases regarding the neighborhood of the vertex v_3 in G and show that if $v_3 \notin L$ then each one comes to a contradiction.

(D.a) The v_3 does not have neighbors in G other than v_5 and v_j . If we assume that $v_3 \notin L$, then v_5 has a neighbor in G which is not a neighbor of v_j and, additionally, v_j has a neighbor in G which is not a neighbor of v_5 . Thus, we can have one of the following three cases, each of which comes to a contradiction:

- $v_2 \in N_G[v_5]$ and $v_7 \in N_G[v_j]$. Now, we have that $v_2v_6 \in E(G)$, since otherwise G has a P_6 induced by the vertices $\{v_4, v_2, v_5, v_6, v_7, v\}$. However, in this case v_2 would not be simple in G_2 , where G_2 is the subgraph of G induced by the vertices to the right of v_2 in σ , since $v_7 \in N_G[v_6]$ and $v_7 \notin N_G[v_5]$ and, also, $v_3 \in N_G[v_5]$ and $v_3 \notin N_G[v_6]$. Indeed, it suffices to show that the vertices v_5, v_6, v_7 , and v_3 belong to the induced subgraph G_2 of G .

We know that $v_5, v_3 \in N_G[v_j]$ and, thus, $v_5 > v_i$ and $v_3 > v_i$ since we have assumed that v_j does not have a neighbor v_x , such that $v_x < v_i$. Additionally, from $v_7 \in N_G[v_j]$ it follows that $v_6 \in N_G[v_j]$, since otherwise G has a C_4 induced by the vertices $\{v_j, v_5, v_6, v_7\}$. Therefore, $v_6, v_7 \in N_G[v_j]$ and, thus, $v_i < v_6$ and $v_i < v_7$. Therefore, the vertices v_5, v_6, v_7 , and v_3 belong to the induced subgraph G_2 of G , and thus, the vertex v_2 is not simple in G_2 , which is a contradiction to our assumption that σ is a strong perfect elimination ordering.

- $v_k \notin N_G[v_5]$ and $v_6 \notin N_G[v_j]$. From $v_k \notin N_G[v_5]$ we obtain that $v_2, v_i \notin N_G[v_5]$. In this case G has a P_8 induced by the vertices $\{v_4, v_2, v_k, v_j, v_5, v_6, v_7, v\}$. This path is chordless since G is a chordal graph.
- $v_i \notin N_G[v_5]$ and $v_6 \notin N_G[v_j]$. In this case, we have a P_8 in G induced by the vertices $\{v_4, v_2, v_k, v_j, v_5, v_6, v_7, v\}$; thus, $v_kv_5 \in E(G)$. From $v_i \notin N_G[v_5]$ we obtain that $v_2 \notin N_G[v_5]$ and, thus, $v_6v_k \in E(G)$. Now, G has a 3-sun induced by the vertices $\{v_5, v_k, v_j, v_6, v_i, v_3\}$, since we have assumed that $v_iv_5 \notin E(G)$, $v_6v_j \notin E(G)$, and other edges do not exist by assumption. This is a contradiction to our assumption that G is a strongly chordal graph.

Using similar arguments as in Case (B.a)(ii) and Case (C.a), we can prove that if $v_3 \notin L$ we come to a contradiction, even in the case where v_j has another neighbor v_x in G_i such that $v_xv_5 \notin E(G)$. Indeed, in the case where $v_2v_5 \in E(G)$ we can prove that $N_G[v_5] \supset N_G[v_j]$ and, thus, $v_3 \in L$. In the case where $v_6v_j \notin E(G)$, the vertex v_x will be simple after the first iteration during the construction of σ and, thus, $v_x < v_y \leq v_i$.

- (D.b) The vertex v_3 has two neighbors v_5 and v'_5 in G_y , such that $v_5v'_5 \notin E(G)$. Using the same arguments as in Case (B.b), we obtain that in this case G has a C_4 which is a contradiction to our assumptions.
- (D.c) The vertex v_3 has two neighbors v_5 and v'_5 (where $v_5 \neq v_j$ and $v'_5 \neq v_j$) in G_y , such that $v_5v'_5 \in E(G)$, and neither $N_y[v_5] \subseteq N_y[v'_5]$ nor $N_y[v'_5] \subseteq N_y[v_5]$. Using the same arguments as in Cases (B.c) and (C.c), we can prove that this case comes to a contradiction.

Case 2: The vertex $v_i \in I$ and $v_i \notin S$. Since σ is a strong perfect elimination ordering, each vertex $v_i \in I$ is simple in G_i and, thus, $\{N_{G_i}[v_k] : v_k \in N_{G_i}[v_i]\}$ is linearly ordered by inclusion. We will show that $\{N_G[v_k] : v_k \in N_{G_i}[v_i] \text{ and } \kappa(v_k) = \kappa(v_i)\}$ is linearly ordered by inclusion for all vertices $v_i \in I$ and $v_i \notin S$. Since v_i is not a simplicial vertex in G , there exist at least two vertices $v'_2, v'_j \in N_G(v_i)$ such that $v'_2v'_j \notin E(G)$. In the case where there exist no neighbors v'_2 and v'_j of v_i , such that $v'_2 < v_i < v'_j$ and $v'_2v'_j \notin E(G)$, we have exactly the same situation as in Case 1, where every neighbor v'_j of v_i in G_i was joined by an edge with every neighbor v'_2 of v_i , such that $v'_2 < v_i < v'_j$. Let us now consider the case where v_i has two neighbors v'_2 and v'_j , such that $v'_2 < v_i < v'_j$ and $v'_2v'_j \notin E(G)$.

Using the same arguments as in Case 1 we can prove that for any vertex $v'_i \in I$ and $v'_i \notin S$, the set $\{N_G[v'_k] : v'_k \in N_{G'_i}[v'_i] \text{ and } \kappa(v'_k) = \kappa(v'_i)\}$ is linearly ordered by inclusion. First, we can easily see that for any two neighbors v'_k and v'_j of v'_i in G'_i , such that $v'_i < v'_k < v'_j$ and $\kappa(v'_i) = \kappa(v'_k) = \kappa(v'_j)$, we can prove that either $N_G[v'_k] \subseteq N_G[v'_j]$ or $N_G[v'_k] \supseteq N_G[v'_j]$, by substituting v_k by v'_k and v_j by v'_j in the proof of Case 1. Additionally, we can see that for any neighbor v'_k of v'_i in G'_i , such that $v'_i < v'_k$ and $\kappa(v'_i) = \kappa(v'_k)$, we can prove that either $N_G[v'_k] \subseteq N_G[v'_i]$ or $N_G[v'_k] \supseteq N_G[v'_i]$, by substituting v_k by v'_i and v_j by v'_k in the proof of Case 1. It is easy to see that by combining these two results we obtain that the set $\{N_G[v'_k] : v'_k \in N_{G'_i}[v'_i] \text{ and } \kappa(v'_k) = \kappa(v'_i)\}$ is linearly ordered by inclusion, for any vertex $v'_i \in I$ and $v'_i \notin S$.

From Cases 1 and 2 we conclude that using the constructed strong perfect elimination ordering σ of G , we have proved that the set $\{N_G[v_k] : v_k \in N_{G_i}[v_i] \text{ and } \kappa(v_k) = \kappa(v_i)\}$ is linearly ordered by inclusion, for any vertex $v_i \in I$. Thus, the lemma holds. ■

From Corollary 2.1, we have that $\lambda(G) \geq \alpha(G)$ holds for any graph G . Since κ is a linear coloring of G using $\alpha(G)$ colors, it follows that the equality $\lambda(G) = \alpha(G)$ holds for G . Since every induced subgraph of a strongly chordal graph is strongly chordal [11], we can construct a strong perfect elimination ordering σ as described above for every induced subgraph G_A of G , $\forall A \subseteq V(G)$; thus, we can assign a coloring κ to G_A with $\alpha(G_A)$ colors. Concluding, the equality $\lambda(G_A) = \alpha(G_A)$ holds for every induced subgraph G_A of a strongly chordal graph G and, therefore, any strongly chordal graph G is a linear graph.

Therefore, we have proved the following result.

Lemma 4.2. *Any P_6 -free strongly chordal graph is a linear graph.*

From Lemma 4.2, we obtain the following result.

Lemma 4.3. *If G is a k -sun graph ($k \geq 3$), then G is a linear graph.*

Proof. Let G be a k -sun graph. It is easy to see that the equality $\alpha(G) = \lambda(G)$ holds for the k -sun G . Since a k -sun constitutes a minimal forbidden subgraph for the class of strongly chordal graphs, it follows that every induced subgraph of a k -sun is a strongly chordal graph, and, thus, from Lemma 4.2 G is a linear graph. ■

From Lemmas 4.2 and 4.3, we also derive the following results.

Proposition 4.5. *Linear graphs form a superclass of the class of P_6 -free strongly chordal graphs.*

We have proved that any P_6 -free chordal graph which is not a linear graph has a k -sun as an induced subgraph; however, the k -sun itself is a linear graph. The interest of these results lies on the following characterization that we obtain for the class of linear graphs in terms of forbidden induced subgraphs.

Theorem 4.2. *Let \mathcal{F} be the family of all the minimal forbidden induced subgraphs of the class of linear graphs, and let F_i be a member of \mathcal{F} . The graph F_i is either a C_n ($n \geq 4$), or a P_6 , or it properly contains a k -sun ($k \geq 3$) as an induced subgraph.*

5 Concluding Remarks

In this paper we introduced the linear coloring on graphs and defined two classes of perfect graphs, which we called co-linear and linear graphs. An obvious though interesting open question is whether combinatorial and/or optimization problems can be efficiently solved on the classes of linear and co-linear graphs. In addition, it would be interesting to study the relation between the linear chromatic number and other coloring numbers such as the harmonious number and the achromatic number on classes of graphs, and also investigate the computational complexity of the the harmonious coloring problem and pair-complete coloring problem on the classes of linear and co-linear graphs.

It is worth noting that the harmonious coloring problem is of unknown computational complexity on co-linear and connected linear graphs, since it is polynomial on threshold and connected quasi-threshold graphs and NP-complete on co-chordal, chordal and disconnected quasi-threshold graphs; note that the NP-completeness results have been proven on the classes of split and interval graphs [1]. However, the pair-complete coloring problem is NP-complete on the class of linear graphs, since its NP-completeness has been proven on quasi-threshold graphs, but it is polynomially solvable on threshold graphs [2], and of unknown complexity on co-chordal and co-linear graphs. Moreover, the Hamiltonian path and circuit problems are NP-complete on the class of linear graphs, since their NP-completeness has been proven on the class of split strongly chordal graphs [19]. We point out that, the complexity status of the path cover problem is open on the class of co-linear graphs.

Finally, it would be interesting to study structural and recognition properties of linear and co-linear graphs and see whether they can be characterized by a finite set of forbidden induced subgraphs.

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